

On Hybrid (Topologically) Massive Supergravity in Three Dimensions

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ABSTRACT

A class of hybrid (topologically) massive off-shell supergravities coupled to an on-shell matter scalar multiplet was recently constructed. The auxiliary field in the off-shell multiplet is dynamical for generic values of the eight parameters. We find that by choosing the parameters appropriately, it remains non-dynamical. We perform linearized analysis around the supersymmetric AdS_3 vacuum and its Minkowski limit. The ghost-free condition for the Minkowski vacuum is explored. For the AdS_3 vacuum, we obtain the criticality condition and find that at the critical points, one of the two massive gravitons becomes pure gauge and decouples from the bulk physics, whilst the other has positive energy. We demonstrate that the mass of the BTZ black hole is non-negative at the critical points. We also investigate general BPS solutions. For certain parameter choices, we obtain exact solutions. In particular, we present the BPS string (domain-wall) solution that is dual to certain two-dimensional quantum field theory with an ultra-violet conformal fixed point.

1 Introduction

The foundation of Einstein's theory of gravity is the principle of the general coordinate transformation invariance. The Einstein-Hilbert action is the minimal dynamical theory that incorporates this principle. Its successes in large scale physics notwithstanding, the theory is non-renormalizable in the frame work of quantum mechanics. Insisting on this principle, the Einstein-Hilbert action can only be modified by adding higher-derivative terms, such as polynomial invariants constructed from the Riemann curvature tensor. While indeed theories with a finite number of such higher-order terms can become renormalizable, it is at the price of unitarity [1]. At the level of effective field theory, string provides an infinite number of higher-order terms in such a specific way that the resulting theory is expected to be both unitary and finite. However, this theory is too complicated to play with.

Gravity in three dimensions is simple since the Einstein-Hilbert action provides no physical degrees of freedom. However, by adding higher derivative terms dynamics can be generated. The resulting theory becomes non-trivial but simpler than Einstein gravity in four dimensions, thus provides a non-trivial toy model for studying quantum gravity. The best known example is topologically massive gravity, which is constructed by adding a Lorentz-Chern-Simons term to the Einstein-Hilbert action. Consequently, a massive graviton emerges [2, 3]. The theory is not unitary with the standard Einstein-Hilbert action, but can be made so by reversing the sign of the action. However, when coupled to a cosmological constant [4], the negative sign of the Einstein-Hilbert action implies that the BTZ black hole, which is an excited state in the theory, has negative mass. In [5], cosmological topologically massive gravity with the standard Einstein-Hilbert action was revisited. It was argued that the ghost-like massive graviton decouples at a certain critical point of the parameter space. The critical theory is then conjectured to be self-consistent quantum gravity *via* the AdS/CFT correspondence.

Subsequently, new massive three-dimensional gravity (NMG) with quadratic Riemann curvature invariants was constructed and shown to be unitary [6]. This inspires later constructions of more general (topologically) massive (super)gravities [7, 8] in three dimensions, as well as the higher dimensional generalizations [9]. (See also a recent review [10].) All these theories involve only the metric (or the supergravity multiplet), and the most general construction has seven parameters [8]. Recently, an eight-parameter $\mathcal{N} = 1$ supergravity with a matter scalar multiplet was constructed [11]. The theory is of particular interest since it is hybrid in that the supergravity multiplet is off-shell with higher derivative terms

in the action whilst the matter multiplet is on-shell with at most two derivatives.

An important feature of all these three-dimensional massive supergravities is that the supersymmetry can be off-shell. The consequence is that such a theory can be complete in terms of supersymmetry by adding just any finite number of higher-derivative super invariants. This is very different from string theory or M-theory whose supersymmetry is realised on-shell: the completeness of the supersymmetry alone requires an infinite number of higher-order terms once just one term beyond the second derivative is introduced. (See for example [12].) Furthermore, the existence of the hybrid theory [11] demonstrates that in three dimensions, even if higher-derivative terms in gravity sector are inevitable in a quantum theory, the matter sector can still be minimally coupled (In stringy frame). This is desirable since the matter sector such as the Standard Model is indeed renormalizable without having to go beyond two derivatives. These features make three dimensions an attractive starting point to study quantum gravity.

We begin our discussion in section 2 by reviewing hybrid topologically massive supergravity. We give the bosonic action, the equations of motion and the supersymmetry transformation rules. As in typical off-shell supergravity, the auxiliary field can acquire dynamics when higher-order super invariants are added to the action. This makes the analysis much more complicated. In section 3, we find that by choosing the eight parameters appropriately, the auxiliary scalar field S stays non-dynamical. This simplifies the theory significantly, but it is non-trivial with five parameters left. Interestingly, it turns out that the ratio of the coefficients of the R^2 and $R^{\mu\nu}R_{\mu\nu}$ terms is $-3/8$, the number needed for the new massive gravity [6] to be unitary. We also present an analogous four-parameter pure massive supergravity without the matter multiplet.

For theories with higher derivative terms, it is typical that massive modes with negative energy can arise. Since the matter multiplet does not involve higher derivative terms, the problem of having negative energy modes lies mainly in the gravity sector. In particular the analysis for the traceless spin-2 graviton mode is identical to that for the pure massive supergravity. Thus our analysis and many conclusions applies for the four-parameter pure massive supergravity.

We first consider the limit of the five-parameter hybrid theory for which its supersymmetric vacuum becomes Minkowski space-time, and then perform a linearized analysis around this vacuum. The spectrum consists of one massless scalar mode and two massive graviton modes. The consequence of the inclusion of the scalar multiplet is that the theory in general becomes non-unitary regardless of the sign of the Einstein-Hilbert action.

However, we find that there exists a special point in parameter space for which the on-shell Hamiltonian for the two massive graviton modes vanishes identically, signalling that the linearized analysis breaks down, and the possibility that the theory may become unitary.

We also perform a linearized analysis around the supersymmetric AdS_3 vacuum. Some aspects of the linear analysis for the tensorial and scalar modes for general parameters were given in [11]. Specializing to our choice of parameters, we find that the theory in general has one scalar mode and two massive graviton modes. Since the matter scalar mode cannot be gauged away, we have to restrict parameters so that the scalar mode is not ghost-like. This can be achieved by requiring that the super invariant in the action involving the Einstein-Hilbert term has to have positive coefficient. (This provides an extra constraint than the pure gravity theory.) Then one of the massive gravitons becomes inevitably ghost-like. We find the critical points for which the ghost massive graviton becomes pure gauge and decouples from the bulk physics and then verify that the remaining massive graviton indeed has positive energy. This result applies also to the four-parameter pure massive supergravity. Owing to the complexity arising from the dynamical nature of the auxiliary field S for generic parameters, no conclusion was made in [11] about the stability of the scalar mode. For our specialized theory where S is non-dynamical, we demonstrate that the scalar mode is stable at the critical points satisfying the Breitlohner-Freedman bound. This suggests that the theory may be well-defined at these critical points.

In section 4, we obtain the mass and angular momentum for the BTZ black hole that is asymptotic to the supersymmetric AdS_3 vacuum. We demonstrate that at the critical points, the mass is non-negative and always greater or equal to the angular momentum. We also verify that the first law of thermodynamics holds.

It should be emphasized that although our analysis focused on the hybrid theory, the linearized analysis of the traceless graviton mode and the BTZ energy calculation apply equally well to the four-parameter pure massive gravity. The critical points for the general massive pure supergravity were obtained in [8]. Our examination of the energy of the remaining non-trivial massive graviton and the mass of the BTZ black hole reveals additional properties of the theory at the critical points. The inclusion of the on-shell matter multiplet does not alter these results, but instead provides further constraints on the parameter space.

In section 5, we study BPS string (domain-wall) solutions of the eight-parameter supergravity. The equations reduce to one third-order ordinary non-linear differential equation. For certain parameter choices, including the five-parameter theory, the solutions can be obtained explicitly. These solutions are asymptotic to AdS_3 and are expected to be dual to

certain two-dimensional field theory with an ultra-violet conformal fixed point. We discuss the characteristics of the spectrum using the standard free-scalar approach.

In section 6, we investigate general BPS solutions. We find that for general choice of the parameters, equations can be reduced to two differential equations. For some special choices of parameters, we are able to obtain the exact solutions. In particular we derive solutions that arise at the critical points of the five-parameter theory discussed in section 3. The paper concludes in section 7.

2 The theory

Hybrid $\mathcal{N} = 1$ topologically massive supergravity involves an off-shell supergravity multiplet (e_μ^a, ψ_μ, S) and an on-shell scalar matter multiplet (ϕ, ψ) . The full action up to quartic fermion were given in [11]. For our purpose, we are concerned with the bosonic sector, namely

$$\begin{aligned} I = & \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma e^{-2\phi} [R + 4(\partial\phi)^2 + 4m^2 + 2S^2] + 4\tilde{m}S - 2a(RS + 2S^3) \right. \\ & + \frac{1}{4}\alpha(4R_{\mu\nu}R^{\mu\nu} - R^2 - 8(\partial S)^2 + 12S^4 + 4RS^2) + c(3RS^2 + 10S^4) \\ & \left. + b(R^2 - 16(\partial S)^2 + 12RS^2 + 36S^4) \right] + 2\beta m \mathcal{L}_{\text{LCS}}, \end{aligned} \quad (2.1)$$

where

$$\mathcal{L}_{\text{LCS}} = \frac{1}{2}\epsilon^{\lambda\mu\nu}\Gamma_{\lambda\sigma}^\rho \left(\partial_\mu \Gamma_{\rho\nu}^\sigma + \frac{2}{3}\Gamma_{\mu\tau}^\sigma \Gamma_{\nu\rho}^\tau \right). \quad (2.2)$$

The theory contains eight parameters, including seven continuous ones $(m, \tilde{m}, \alpha, \beta, a, b, c)$ and one discrete σ which takes values of $0, \pm 1$. We do not count the three dimensional κ as a non-trivial parameter. The supersymmetric transformation rules are given by [11]

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, & \delta\psi_\mu &= D_\mu\epsilon + \frac{1}{2}\gamma_\mu S\epsilon, & \delta S &= \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}D_\mu\psi_\nu - \frac{1}{4}\bar{\epsilon}\gamma^\mu\psi_\mu S; \\ \delta\psi &= \frac{1}{4}e^{\frac{5}{4}\phi}(\gamma^\mu\partial_\mu\phi + S + m)\epsilon, & \delta\phi &= e^{-\frac{5}{4}\phi}\bar{\epsilon}\psi. \end{aligned} \quad (2.3)$$

The first three transformations are self contained and close off-shell; the last two transformations close on-shell. As discussed in [11], one cannot truncate out the scalar multiplet to obtain the seven-parameter theory. The supersymmetry transformation rules imply that setting (ϕ, ψ) zero, the auxiliary field S is fixed and the whole theory reduces to standard cosmological topologically massive supergravity[13].

The equation of motion associated with variation of the dilaton is given by

$$4\Box\phi - 4(\partial\phi)^2 + R + 2S^2 + 4m^2 = 0. \quad (2.4)$$

The equation of motion for the auxiliary field S is given by

$$(\alpha + 8b)\square S + \sigma e^{-2\phi}S + \tilde{m} - \frac{1}{2}a(R + 6S^2) + (3\alpha + 36b + 10c)S^3 + \frac{1}{2}(\alpha + 12b + 3c)RS = 0. \quad (2.5)$$

The Einstein's equations are more complicated, given by

$$\begin{aligned} & \sigma e^{-2\phi}(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi) - 2\tilde{m}Sg_{\mu\nu} \\ & + \alpha \left[\square R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R - 4R_\mu{}^\lambda R_{\lambda\nu} + \frac{5}{2}RR_{\mu\nu} + \frac{3}{2}g_{\mu\nu}(R_{\rho\sigma}R^{\rho\sigma} - \frac{7}{12}R^2) \right. \\ & \quad \left. - \frac{3}{2}S^4g_{\mu\nu} + G_{\mu\nu}S^2 - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)S^2 - 2\partial_\mu S\partial_\nu S + (\partial S)^2g_{\mu\nu} \right] \\ & - 2\beta mC_{\mu\nu} + 2a \left[S^3g_{\mu\nu} - G_{\mu\nu}S + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)S \right] \\ & - b \left[(\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)F - FR_{\mu\nu} + 16\partial_\mu S\partial_\nu S + \frac{1}{2}g_{\mu\nu}(R^2 - 16(\partial S)^2 + 12RS^2 + 36S^4) \right] \\ & + c \left[3S^2R_{\mu\nu} - 3(\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)S^2 - \frac{1}{2}g_{\mu\nu}(3RS^2 + 10S^4) \right] = 0, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \\ C_{\mu\nu} &= \epsilon_\mu{}^{\rho\sigma}\nabla_\rho(R_{\sigma\nu} - \frac{1}{4}g_{\sigma\nu}R), \\ F &= 2(R + 6S^2). \end{aligned} \quad (2.7)$$

Note that the supergravity multiplet involves up to four derivatives whilst the matter scalar ϕ involves at most two derivatives.

3 Super-NMG with scalar multiplet and CS terms

As we can see in the previous section, the auxiliary field S in the generalized topologically massive gravity acquires dynamical terms when higher-order off-shell super invariants are involved. However, for some specific choice of parameters, we find that the S field remains non-dynamical. This corresponds to set

$$\alpha = 6c, \quad a = 0, \quad b = -\frac{3}{4}c. \quad (3.1)$$

The reduced theory has five parameters, and the action is given by

$$\begin{aligned} I &= \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma e^{-2\phi}(R + 4(\partial\phi)^2 + 4m^2 + 2S^2) + 4\tilde{m}S + \frac{1}{6\nu^2}S^4 \right. \\ & \quad \left. + \frac{1}{\nu^2}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2) \right] + \frac{1}{\mu}\mathcal{L}_{\text{LCS}}. \end{aligned} \quad (3.2)$$

Note that here we have renamed two parameters, namely $\nu^2 = 1/(6c)$ and $\mu = 1/(2\beta m)$. The parameters (m, \tilde{m}, μ, ν) all have the same dimension $[\text{length}]^{-1}$. The parameter ν is

chosen for the convenience of dimensional analysis and it is understood that ν^2 can be negative as well.

The equation of motion for S now becomes purely algebraic, namely

$$\sigma S e^{-2\phi} + \tilde{m} + \frac{1}{6\nu^2} S^3 = 0. \quad (3.3)$$

As in the general case, the supersymmetric vacuum is an AdS_3 [11], but now with

$$R_{\mu\nu} = -2m^2 g_{\mu\nu}, \quad S = -m, \quad \phi = 0, \quad \tilde{m} = -\sigma m - \frac{m^3}{6\nu^2}. \quad (3.4)$$

Note that in this paper, we shall take a convention that $\phi = 0$ for the AdS_3 vacuum. This can always be achieved if we let the parameter σ to be continuous. When $m = 0 = \tilde{m}$, the vacuum becomes Minkowski space-time.

It is interesting to note that the $-\frac{3}{8}$ factor of ghost-free new massive gravity [6] also arises in our case. Thus the theory is a hybrid generalization of the pure super-NMG. It is worth pointing out that all the off-shell super invariants in (2.1) that decouple from the dilaton ϕ are the exactly the same as those in pure massive supergravity constructed in [8]. This implies that the specialization we obtain in our hybrid theory also exists in the pure supergravity theory. It is given by

$$\begin{aligned} I = & \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma(R - 2S^2) + 4\tilde{m}S + \frac{1}{6\nu^2} S^4 \right. \\ & \left. + \frac{1}{\nu^2} (R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2) \right] + \frac{1}{\mu} \mathcal{L}_{\text{LCS}}. \end{aligned} \quad (3.5)$$

The theory has four non-trivial parameters. The fermionic action can be read off from [8]. The off-shell supersymmetric transformation rule is given by

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, \quad \delta \psi_\mu = D_\mu \epsilon + \frac{1}{2} \gamma_\mu S \epsilon, \quad \delta S = \frac{1}{4} \bar{\epsilon} \gamma^{\mu\nu} D_\mu \psi_\nu - \frac{1}{4} \bar{\epsilon} \gamma^\mu \psi_\mu S. \quad (3.6)$$

It should be emphasized that this pure super-NMG cannot be obtained by truncating out the the scalar multiplet from (3.2).

Although our primary focus is on the hybrid theory (3.2), many of the results can also apply to the pure gravity theory (3.5). This is because the scalar matter mulitplet involves only two derivatives and hence there is no massive mode with negative energy associated with the scalar multiplet. For the tensorial modes, as we shall discuss later, with appropriate gauge choice, the linear analysis is exact the same for both theories.

3.1 The Minkowski limit

Let us first set $m = 0 = \tilde{m}$ so that the supersymmetric vacuum is three-dimensional Minkowski space-time. We consider linearized excitations around the vacuum

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad S = \bar{S} + s, \quad \phi \rightarrow \bar{\phi} + \phi, \quad (3.7)$$

where $\bar{S} = 0 = \bar{\phi}$. We may adopt the following gauge for the metric [7, 15]

$$h_{ij} = -\varepsilon^{ik}\varepsilon^{jl}\frac{\partial_k\partial_l}{\nabla^2}\varphi, \quad h_{0i} = -\varepsilon^{ij}\frac{1}{\nabla^2}\partial_j\xi, \quad h_{00} = \frac{1}{\nabla^2}(N + \square\varphi). \quad (3.8)$$

This gauge choice amounts to

$$\partial_i h_{ij} = 0, \quad \partial_i h_{0i} = 0. \quad (3.9)$$

It should be emphasized that in this special gauge, (φ, ξ, N) are exactly the three gauge-invariant quantities constructed from the metric.

The quadratic action for perturbations around the vacuum is given by

$$I^{(2)} = \int d^3x \left\{ -\sigma \left(\frac{1}{2}(N\varphi + \varphi\square\varphi - \xi^2) + 4\phi\square\phi + 2\phi N + 4\phi\square\varphi \right) - \frac{1}{2\mu}N\xi + \frac{1}{2\nu^2}\xi\square\xi + \frac{1}{8\nu^2}N^2 \right\}. \quad (3.10)$$

The equation of motion for N is purely algebraic and can be solved explicitly, given by

$$N = 4\nu^2 \left(\frac{1}{2\mu}\xi + 2\sigma\phi + \frac{1}{2}\sigma\varphi \right). \quad (3.11)$$

Sustituting this into (3.10), we have

$$\begin{aligned} I^{(2)} &= \int d^3x (T - V), \\ T &= -\sigma \left(\frac{1}{2}\varphi\square\varphi + 4\phi\square\phi + 4\phi\square\varphi \right) + \frac{1}{2\nu^2}\xi\square\xi, \\ V &= -\frac{1}{2}\sigma\xi^2 + 2\nu^2 \left(\frac{1}{2\mu}\xi + 2\sigma\phi + \frac{1}{2}\sigma\varphi \right)^2. \end{aligned} \quad (3.12)$$

It is clear that if we diagonalize the kinetic terms of (ϕ, φ) , there is a ghost field regardless the sign of the parameter σ . Since the kinetic terms always involve both ghost and non-ghost fields, we cannot diagonalize the kinetic and mass terms simultaneously. Nevertheless we can analyze its spectrum by examining the equations of motion. They are given by

$$\begin{aligned} 2\square\phi + \square\varphi + 2\nu^2 \left(\frac{1}{2\mu}\xi + 2\sigma\phi + \frac{1}{2}\sigma\varphi \right) &= 0, \\ 4\square\phi + \square\varphi + 2\nu^2 \left(\frac{1}{2\mu}\xi + 2\sigma\phi + \frac{1}{2}\sigma\varphi \right) &= 0, \\ \frac{1}{\nu^2}\square\xi + \left(\sigma - \frac{\nu^2}{\mu} \right)\xi - \frac{2\nu^2\sigma}{\mu}(2\phi + \frac{1}{2}\varphi) &= 0. \end{aligned} \quad (3.13)$$

In the momentum space

$$\varphi = \int d^3\vec{p}(\varphi_p e^{-ipx} + c.c.), \quad \phi = \int d^3\vec{p}(\phi_p e^{-ipx} + c.c.), \quad \xi = \int d^3\vec{p}(\xi_p e^{-ipx} + c.c.), \quad (3.14)$$

the equations of motion become

$$\begin{aligned} M^2(2\phi_p + \varphi_p) + 2\nu^2\left(\frac{1}{2\mu}\xi_p + 2\sigma\phi_p + \frac{1}{2}\sigma\varphi_p\right) &= 0, \\ M^2(\varphi_p + 4\phi_p) + 2\nu^2\left(\frac{1}{2\mu}\xi_p + 2\sigma\phi_p + \frac{1}{2}\sigma\varphi_p\right) &= 0, \\ \left(\frac{M^2}{\nu^2} + \sigma - \frac{\nu^2}{\mu^2}\right)\xi_p - \frac{2\nu^2\sigma}{\mu}(2\phi_p + \frac{1}{2}\varphi_p) &= 0. \end{aligned} \quad (3.15)$$

where $M^2 \equiv (p^0)^2 - \vec{p} \cdot \vec{p}$ is the mass-squared parameter. There are three non-trivial solutions. The first has vanishing M with

$$\varphi_p = -4\phi_p, \quad \xi_p = 0. \quad (3.16)$$

The corresponding on-shell Hamiltonian is given by

$$H = 4\sigma \int d^3x (\dot{\phi}^2 + (\nabla\phi)^2). \quad (3.17)$$

Thus $\sigma = -1$ gives rise to a massless ghost scalar. The absence of such a ghost requires that $\sigma \geq 0$.

The other two solutions are given by

$$M_{\pm}^2 = \frac{\nu^2}{2} \left[\frac{\nu^2}{\mu^2} - 2\sigma \pm \sqrt{\frac{\nu^2}{\mu^2} \left(\frac{\nu^2}{\mu^2} - 4\sigma \right)} \right], \quad \phi_p = 0, \quad \xi_p = -\frac{\mu}{\nu^2} (M_{\pm}^2 + \sigma\nu^2) \varphi_p. \quad (3.18)$$

The absence of tachyon modes is guaranteed by

$$\sigma > 0, \quad \nu^2 \in (-\infty, 0) \cup (4\sigma\mu^2, +\infty) \quad \text{or} \quad \sigma < 0, \quad \nu^2 \in (-\infty, 4\sigma\mu^2) \cup (0, +\infty). \quad (3.19)$$

In the limit $\nu \rightarrow \infty$, M_{\pm} becomes infinity and the corresponding mode decouples. In addition, $M_{-} \rightarrow (\sigma\mu)^2$ and hence the corresponding mode is exactly the massive graviton discussed in [2]. At the first sight, the massive modes are scalars, but a careful study of the supersymmetry transformation rules shows that they are in fact spin-2 particles [7, 8]. Thus the spectrum consists of one massless scalar modes and two massive graviton modes.

The on-shell Hamiltonian for the massive gravitons are given by

$$\begin{aligned} H_{\pm} &= \frac{1}{2} \left(\frac{\mu^2}{\nu^6} (M_{\pm}^2 + \sigma\nu^2)^2 - \sigma \right) (\dot{\varphi}^2 + (\nabla\varphi)^2) \\ &\quad + \left(\frac{M_{\pm}^4}{2\nu^2} - \frac{\sigma\mu^2}{2\nu^4} (M_{\pm}^2 + \sigma\nu^2)^2 \right) \varphi^2. \end{aligned} \quad (3.20)$$

It is easy to see that for $\sigma < 0$, both kinetic terms are non-negative. The theory has one ghost-like massless scalar and two ghost-free massive gravitons. For $\sigma > 0$, the kinetic term in H_- is negative, and the theory has a well-defined massless scalar, but one of the two gravitons is ghost like. Note that for pure new massive supergravity with no matter scalar multiplet, the theory is ghost free when $\sigma < 0$.

When the condition (3.19) is saturated, namely

$$\nu^2 = 4\sigma\mu^2, \quad (3.21)$$

both Hamiltonian H_{\pm} vanish, signaling that the linearized analysis breaks down and suggesting a possibility that the theory might become ghost free. Of course, to determine this definitively, higher-order interactions become non-negligible and a non-perturbative analysis may be necessary. We shall not proceed in this direction here.

3.2 Linearization around the AdS_3

The linear analysis in Minkowski space-time demonstrates that there is at least one ghost-like field in the spectrum. It is then of interest to study the linear perturbation in the supersymmetric AdS_3 to investigate whether there exists a critical point where this ghost field decouples from the bulk physics, as in chiral gravity proposed in [5]. Since the matter scalar ϕ has the standard dynamics, there can be no critical points associated with the scalar and its physical degree of freedom cannot be gauged away. It is thus necessary to require that $\sigma > 0$. As we see in the flat space-time analysis, the choice of $\sigma > 0$ implies that one of the two massive gravitons is ghost like. We are interested in finding critical points to gauge away this ghost graviton, but still to keep the other one so that bulk gravity is non-trivial.

The linear perturbation of the eight-parameter theory (2.1) was analyzed in [11]. Owing to the complexity of the theory and the dynamical nature of the auxiliary field S , there was not a concrete conclusion for the scalar perturbation. The situation becomes much simpler for the reduced action (3.2) for which the auxiliary field S is non-dynamical.

As in [11], we expand the metric around the supersymmetric AdS_3 background (3.4) as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and impose the gauge condition

$$\nabla^\mu (h_{\mu\nu} - \frac{1}{3}\bar{g}_{\mu\nu}h) = 0, \quad \text{where} \quad h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}. \quad (3.22)$$

The scalar fields (S, ϕ) are expanded around the symmetric solution $\bar{S} = -m$ and $\bar{\phi} = 0$, and we denote the fluctuation fields by s and ϕ , respectively. Setting $m = 1$ for simplicity, the equations for the scalar modes are given by

$$6\Box\phi - (\Box - 3)h - 6s = 0,$$

$$\begin{aligned}
(\sigma + \frac{1}{2\nu^2})s + 2\sigma\phi &= 0, \\
(\frac{1}{2\nu^2} - \sigma)(\square - 3)h + 12\sigma(\square - 3)\phi &= 0.
\end{aligned} \tag{3.23}$$

Note that the parameter μ does not enter the equations of the scalar modes. Depending on the relation between the parameters σ and ν , four classes of solutions may emerge.

Firstly, when $\sigma = -1/(2\nu^2)$, the equations in (3.23) can be reduced to

$$(\square - 3)h = 0, \quad \phi = 0, \quad s = 0. \tag{3.24}$$

This is exactly the same as the scalar perturbation of the pure AdS₃ gravity, and hence it can be easily shown to be a pure gauge. Secondly, for $\sigma = 1/(2\nu^2)$, (3.23) reduces to

$$(\square - 3)h - 24\phi = 0, \quad (\square - 3)\phi = 0. \tag{3.25}$$

The third class corresponds to $(\square - 3)h = 0$ and $(\square - 3)\phi = 0$. This requires that $\sigma = -3/(10\nu^2)$ and $s = 3\phi$. Although h is pure gauge in this case, the free scalar ϕ is non-trivial. The remaining fourth class corresponds to generic values of σ and μ . Since we are looking for solutions where h is non-trivial, the last equation in (3.23) implies that

$$h = \frac{24\sigma\nu^2}{2\sigma\nu^2 - 1}\phi. \tag{3.26}$$

We then find

$$\left(\square - \frac{8\sigma\nu^2(1 + 4\sigma\nu^2)}{(1 + 2\sigma\nu^2)^2}\right)\phi = 0, \quad s = \frac{-4\sigma\nu^2}{2\sigma\nu^2 + 1}\phi. \tag{3.27}$$

The Breitenlohner-Freedman bound in 3-dimensions gives

$$\frac{8\sigma\nu^2(1 + 4\sigma\nu^2)}{(1 + 2\sigma\nu^2)^2} \geq -1 \quad \implies \quad \nu^2(\sigma + 3\sigma^2\nu^2) \geq -\frac{1}{12}. \tag{3.28}$$

The equations for the tensor modes are determined by the traceless part of the Einstein equations. It was shown in [11] that these equations are the same as the seven-parameter pure gravity theory constructed in [8]. Thus we shall just present the result here, but specializing to our specific choice of parameters. For those interested in a detailed analysis, we refer to a recent paper [10]. In the case of $\gamma \equiv \sigma - 1/(2\nu^2) \neq 0$, the transverse traceless massive graviton modes satisfy

$$D(\eta_{\pm})h_{\mu\nu} = 0, \quad D(\eta)_{\mu}^{\nu} = \delta_{\mu}^{\nu} + \eta\varepsilon_{\mu}^{\alpha\nu}\bar{\nabla}_{\alpha}, \tag{3.29}$$

with

$$\eta_{\pm} = \gamma^{-1}\left(\frac{1}{2\mu} \pm \sqrt{\frac{1}{4\mu^2} - \frac{\gamma}{\nu^2}}\right). \tag{3.30}$$

Note that in the limit of $\mu^2 \rightarrow \infty$, corresponding to turning off the Lorentz-Chern-Simons term, we need require that $\gamma/\nu^2 < 0$. In the limit of $\nu^2 \rightarrow \infty$, we have

$$\eta_{\pm} = \frac{1 + \mu/|\mu|}{2\mu\sigma} = 0, \quad \text{or} \quad \frac{1}{\sigma\mu}. \quad (3.31)$$

As we shall see later, the mode with $\eta = 0$ becomes infinitely massive and decouples from the spectrum. The unitarity of the dual CFT requires that

$$|\eta_{\pm}| \leq 1. \quad (3.32)$$

The theory becomes critical when we have $|\eta_+| = 1$ or $|\eta_-| = 1$. This can be achieved by

$$\sigma = -\frac{1}{2\nu^2} + \frac{1}{\mu} \quad \text{or} \quad \sigma = -\frac{1}{2\nu^2} - \frac{1}{\mu}. \quad (3.33)$$

The central charges for the right-handed and left-handed Virasoro algebra of the boundary CFT can be obtained [16]-[22], given by

$$C_{L,R} = \frac{3}{2G_3}(\sigma + \frac{1}{2\nu^2} \mp \frac{1}{\mu}). \quad (3.34)$$

Thus we see that the either C_L or C_R vanishes at the critical points. To be specific, we summarize the four critical points as follows:

$$\begin{aligned} \text{case 1 : } & \sigma = \frac{1}{\mu} - \frac{1}{2\nu^2}, \quad \mu > 0, \quad \nu^2 \geq 2\mu, \\ & \eta_+ = 1, \quad 0 \leq \eta_- = \frac{\mu}{\nu^2 - \mu} \leq 1, \quad C_L = 0, \quad C_R = \frac{2}{\mu}; \\ \text{case 2 : } & \sigma = \frac{1}{\mu} - \frac{1}{2\nu^2}, \quad \mu > 0, \quad \nu^2 < 0, \\ & \eta_+ = 1, \quad -1 < \eta_- = \frac{\mu}{\nu^2 - \mu} \leq 0, \quad C_L = 0, \quad C_R = \frac{2}{\mu}; \\ \text{case 3 : } & \sigma = -\frac{1}{\mu} - \frac{1}{2\nu^2}, \quad \mu < 0, \quad \nu^2 < 0, \\ & 0 \leq \eta_+ = \frac{\mu}{\nu^2 + \mu} < 1, \quad \eta_- = -1, \quad C_L = -\frac{2}{\mu}, \quad C_R = 0, \\ \text{case 4 : } & \sigma = -\frac{1}{\mu} - \frac{1}{2\nu^2}, \quad \mu < 0, \quad \nu^2 \geq -2\mu, \\ & -1 \leq \eta_+ = \frac{\mu}{\nu^2 + \mu} \leq 0, \quad \eta_- = -1, \quad C_L = -\frac{2}{\mu}, \quad C_R = 0. \end{aligned} \quad (3.35)$$

In all the above four critical points, the Breitlohner-Freedman condition (3.28) for the scalar modes and the CFT unitarity conditions (3.32) are satisfied. Furthermore, the σ in all these cases are positive definite implying that the matter scalar ϕ is ghost free.

It is clear that at the critical points, one of the massive graviton modes, corresponding to either $\eta_+ = 1$ or $\eta_- = -1$ and with vanishing associated central charge, becomes pure gauge and decouples from the bulk physics. This feature is the same as chiral gravity [5].

However, there is an important difference that a non-trivial massive graviton mode still survives in our theory at the critical points. It is thus necessary to verify that this mode has positive energy. To compute the energy of the pure graviton mode, we set $\phi = s = 0$, and also $h = 0$. We obtain the quadratic action for the transverse traceless graviton. After integrating by parts, we have

$$I^{(2)} = \frac{1}{2\kappa^2} \int d^3x \sqrt{-\bar{g}} \left\{ \frac{1}{2\nu^2} \bar{\nabla}^2 h^{\mu\nu} \bar{\nabla}^2 h_{\mu\nu} - \frac{1}{2} \left(\sigma + \frac{9m^2}{2\nu^2} \right) \bar{\nabla}^\lambda h^{\mu\nu} \bar{\nabla}_\lambda h_{\mu\nu} \right. \\ \left. + \left(\sigma + \frac{5m^2}{2\nu^2} \right) m^2 h^{\mu\nu} h_{\mu\nu} - \frac{1}{\mu} \varepsilon_\mu^{\alpha\beta} \left(\frac{1}{2} \bar{\nabla}_\alpha h^{\mu\nu} \bar{\nabla}^2 h_{\beta\nu} + m^2 \bar{\nabla}_\alpha h^{\mu\nu} h_{\beta\nu} \right) \right\} \quad (3.36)$$

Note that we restore the parameter m to keep track the dimensions of various terms. In the global coordinates, the metric of AdS_3 vacuum takes the form

$$ds^2 = l^2 (-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2) \quad (3.37)$$

where the radius is related to m by

$$l^2 = m^{-2}. \quad (3.38)$$

It is clear that in this coordinate system, the above action has τ translational invariance, the corresponding Noether charge is given by

$$H = \int d^2x (\pi^{(1)\mu\nu} \dot{h}_{\mu\nu} + \pi^{(2)\mu\nu} \bar{\nabla}_0 \dot{h}_{\mu\nu} - L^{(2)}), \quad (3.39)$$

where

$$\pi^{(1)\mu\nu} = \frac{\sqrt{-\bar{g}}}{2\kappa^2} [\bar{\nabla}^0 \left(-\left(\sigma + \frac{9m^2}{2\nu^2} \right) h^{\mu\nu} + \frac{1}{2\mu} \varepsilon^{\beta\alpha(\mu} \bar{\nabla}_\alpha h_{\beta}^{\nu)} - \nu^{-2} \bar{\nabla}^2 h^{\mu\nu} \right) \\ - \frac{1}{\mu} \varepsilon^{0\beta(\mu} \left(\frac{1}{2} \bar{\nabla}^2 h_{\beta}^{\nu)} + m^2 h_{\beta}^{\nu)} \right)], \\ \pi^{(2)\mu\nu} = \frac{\sqrt{-\bar{g}}}{2\kappa^2} g^{00} [\nu^{-2} \bar{\nabla}^2 h^{\mu\nu} - \frac{1}{2\mu} \varepsilon^{\beta\alpha(\mu} \bar{\nabla}_\alpha h_{\beta}^{\nu)}]. \quad (3.40)$$

Following [5], we identify the conserved charge as the energy of the graviton. Using equations of motion, we find that the above quantities for massive gravitons become

$$\pi_M^{(1)\mu\nu} = -\frac{\sqrt{-\bar{g}}}{2\kappa^2} \left[\left(2m^2 \nu^{-2} + \frac{1}{2\mu\eta_\pm} \right) \bar{\nabla}^0 h_M^{\mu\nu} + \frac{\nu^2}{2\mu} \left(\sigma + \frac{m^2}{2\nu^2} - \frac{1}{\mu\eta_\pm} \right) \varepsilon_\beta^{0\mu} h_M^{\beta\nu} \right], \\ \pi_M^{(2)\mu\nu} = -\frac{\sqrt{-\bar{g}}}{2\kappa^2} g^{00} \left(\sigma + \frac{5m^2}{2\nu^2} - \frac{1}{2\mu\eta_\pm} \right) h_M^{\mu\nu}. \quad (3.41)$$

Specializing to the linearized graviton, upon utilizing their equations of motion, we have the energies

$$H_M = \left(\sigma + \frac{m^2}{2\nu^2} - \frac{1}{\mu\eta_\pm} \right) \int d^2x \frac{\sqrt{-\bar{g}}}{2\kappa^2} (\bar{\nabla}^0 h_M^{\mu\nu} \dot{h}_{M\mu\nu} - \frac{\nu^2}{2\mu} \varepsilon_\beta^{0\mu} h_M^{\beta\nu} \dot{h}_{M\mu\nu}). \quad (3.42)$$

In the limit $\mu \rightarrow \infty$, the energy formula reduces to the one obtained in [23] for new massive gravity. The limit $\nu \rightarrow \infty$ is more subtle. As we see from (3.30) and (3.31), one of the η 's vanishes in this limit, the corresponding energy (3.42) is infinite and hence its mode decouples from the spectrum. For $\eta = 1/(\sigma\mu)$, we need expand η to the next order in $1/\nu^2$, and the resulting energy is then precisely the one given in [5].

The solutions for (3.29) were obtained in [5]. The relevant one corresponding to the primary state is given by the real or imaginary part of $\psi_{\mu\nu}$, where

$$\psi_{\mu\nu} = \frac{e^{-ihu-i\bar{h}v} \sinh^2 \rho}{(\cosh \rho)^{h+\bar{h}}} \begin{pmatrix} 1 & \frac{1}{2}(h-\bar{h}) & \frac{2i}{\sinh 2\rho} \\ \frac{1}{2}(h-\bar{h}) & 1 & \frac{i(h-\bar{h})}{\sinh 2\rho} \\ \frac{2i}{\sinh 2\rho} & \frac{i(h-\bar{h})}{\sinh 2\rho} & -\frac{4}{\sinh^2 2\rho} \end{pmatrix}. \quad (3.43)$$

Here $u = \tau + \phi$ and $v = \tau - \phi$ are the light-cone coordinates. The solution is parameterized by the weights (h, \bar{h}) of the left and right Virasoro algebra of the dual conformal field theory. The weights are given by [5]

$$h = \frac{3}{2} \pm \frac{1}{2\eta}, \quad \bar{h} = -\frac{1}{2} \pm \frac{1}{2\eta}. \quad (3.44)$$

For $\eta \in (0, 1]$, the plus sign is chosen; for $\eta \in [-1, 0)$, the minus sign is chosen. Note that we have set again $m = 1$. At the four critical points discussed earlier, one of the two massive graviton becomes pure gauge, and its energy defined by (3.42) indeed vanishes identically. The energy for the remaining massive graviton, cataloged as in (3.35), is given by

$$\begin{aligned} \text{Case 1 : } \quad H_M(\eta_-) &\propto \frac{2\nu^2(\nu^2 - 2\mu)^2(\mu + \nu^2)}{\mu^3(2\nu^2 - \mu)(\nu^2 - \mu)}, \\ \text{Case 2 : } \quad H_M(\eta_-) &\propto \frac{2(\nu^2 - 2\mu)^2(\nu^4 - \nu^2\mu + 2\mu^2)}{\mu^3(3\mu - 2\nu^2)(\mu - \nu^2)}, \\ \text{Case 3 : } \quad H_M(\eta_+) &\propto \frac{2(\nu^2 - \mu)(\nu^2 + 2\mu)^3}{\mu^3(\nu^2 + \mu)(3\mu + 2\nu^2)}, \\ \text{Case 4 : } \quad H_M(\eta_+) &\propto \frac{2\nu^2(\nu^2 + 2\mu)(\nu^4 + \nu^2\mu + 2\mu^2)}{\mu^3(\nu^2 + \mu)(\mu + 2\nu^2)}. \end{aligned} \quad (3.45)$$

Examining the range of the parameters (μ, ν^2) listed in (3.35), we conclude that the energy for both case 1 and case 2 are non-negative. For the energy of case 3 to be non-negative, a condition $\nu^2 \geq \mu$ must be further imposed. The energy of case 4 is always negative.

Since the analysis of the tensorial modes is the same as that for pure massive supergravities [8, 11], our results also apply to the pure gravity theory (3.5), for which the trace mode h decouples. Although critical points were obtained previously by examining the linearized equations of motion for pure massive supergravity [8], our results provide further details of the on-shell energy for the surviving massive graviton.

Thus we have shown that there exist critical points of the parameters such that one massive graviton becomes pure gauge whilst the other has positive energy and hence the theory is ghost free. This strongly suggests that these critical theories, with or without the matter scalar multiplet, may be well-defined at the full quantum level.

4 Positivity of the BTZ black hole mass

In the previous section, we demonstrate that the hybrid theory with five parameters obtained in section 2 has critical points for which one of the two massive graviton modes becomes trivial and the other has positive energy. In this section, we investigate the energy of the BTZ black hole that is asymptotic to the supersymmetric AdS₃ vacuum. The procedure of calculating the mass and angular momentum of such a black hole in a theory with higher-derivative curvature terms was spelled out in detail in [24]. The modifications to the energy and angular momentum of the BTZ black hole due to the Lorentz Chern-Simons term were obtained in [25]. Here we shall present the formalism of the modifications due to the ν term. For our purpose, we set $\phi = 0$ and $S = -m$. We expand the BTZ black hole around the AdS₃ background as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where

$$\bar{R}_{\mu\nu\alpha\beta} = -\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha}, \quad \bar{R}_{\mu\nu} = -2\bar{g}_{\mu\nu}, \quad \bar{R} = -6. \quad (4.1)$$

Note that we have set $m = 1$ for convenience. Expressing the Einstein equations of motion as $E_{\mu\nu} = 0$, we can expand the equations around the background, giving

$$E_{\mu\nu}^L \equiv \kappa^2 T_{\mu\nu}, \quad (4.2)$$

where L denotes the linear part of the Einstein equations. All the non-linear quantities are lumped together and expressed as $T^{\mu\nu}$. Because of the linearized Bianchi identity $\bar{\nabla}_\mu E^{L\mu\nu} = 0$, $T_{\mu\nu}$ is covariantly conserved with respect to the background metric. Therefore $j^\mu \equiv T^{\mu\nu}\xi_\nu$ is a conserved current, where ξ^μ is a Killing vector in the background metric. The corresponding conserved charge is given by

$$Q(\xi) = \frac{1}{8\pi G_3} \int d\Sigma_\mu j^\mu. \quad (4.3)$$

Here Σ is a two-dimensional space-like hypersurface and $d\Sigma_\mu = n_\mu \sqrt{\gamma} d^2x$ where n_μ is the unit normal vector and γ is the determinant of the induced metric on Σ .

In our case with the σ and ν terms, we have

$$\kappa^2 T_{\mu\nu} = E_{\mu\nu}^L = \left(\sigma + \frac{3}{2\nu^2}\right) \mathcal{G}_{\mu\nu}^L + \frac{1}{4\nu^2} (\bar{g}_{\mu\nu} \bar{\square} - \bar{\nabla}_\mu \bar{\nabla}_\nu - 2\bar{g}_{\mu\nu}) R^L \quad (4.4)$$

$$+ \frac{1}{\nu^2} (\bar{\square} \mathcal{G}_{\mu\nu}^L + \bar{g}_{\mu\nu} R^L) + \frac{1}{\mu\sqrt{-\bar{g}}} \epsilon^{\mu\alpha\beta} \bar{g}_{\beta\sigma} \bar{\nabla}_\alpha \left(R_L^{\sigma\nu} - 2\Lambda h^{\sigma\nu} - \frac{1}{4} \bar{g}^{\sigma\nu} R_L \right), \quad (4.5)$$

where

$$\begin{aligned}
R^L &= -\bar{\square}h + \bar{\nabla}_\mu \bar{\nabla}_\nu h^{\mu\nu} + 2h, \quad h = \bar{g}^{\mu\nu} h_{\mu\nu}, \\
R_{\mu\nu}^L &= \frac{1}{2}(-\bar{\square}h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\rho\mu} + \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\rho\nu}), \\
\mathcal{G}_{\mu\nu}^L &= R_{\mu\nu}^L - \frac{1}{2}\bar{g}_{\mu\nu}R^L + 2h_{\mu\nu}.
\end{aligned} \tag{4.6}$$

We find that j^μ can be expressed as $\nabla_\nu \mathcal{F}^{\mu\nu}$, where

$$\begin{aligned}
\mathcal{F}^{\mu\nu} &= \left(\sigma + \frac{1}{2\nu^2}\right)\{\xi_\alpha \bar{\nabla}^\mu h^{\nu\alpha} - \xi_\alpha \bar{\nabla}^\nu h^{\mu\alpha} + \xi^\mu \bar{\nabla}^\nu h - \xi^\nu \bar{\nabla}^\mu h + h^{\mu\alpha} \bar{\nabla}^\nu \xi_\alpha - h^{\nu\alpha} \bar{\nabla}^\mu \xi_\alpha \\
&\quad + \xi^\nu \bar{\nabla}_\alpha h^{\mu\alpha} - \xi^\mu \bar{\nabla}_\alpha h^{\nu\alpha} + h \bar{\nabla}^\mu \xi^\nu\} + \frac{1}{4\nu^2}\{\xi^\mu \bar{\nabla}^\nu R^L - \xi^\nu \bar{\nabla}^\mu R^L + R_L \bar{\nabla}^\mu \xi^\nu\} \\
&\quad + \frac{1}{\nu^2}\{\xi_\alpha \bar{\nabla}^\nu \mathcal{G}^{L\mu\alpha} - \xi_\alpha \bar{\nabla}^\mu \mathcal{G}^{L\nu\alpha} - \mathcal{G}^{L\mu\alpha} \bar{\nabla}^\nu \xi_\alpha + \mathcal{G}^{L\nu\alpha} \bar{\nabla}^\mu \xi_\alpha\} + \mathcal{F}^{\mu\nu}(\mu),
\end{aligned} \tag{4.7}$$

where the μ -term standing for contribution from Chern-Simons term can be read from the formulae presented in [25]. Using Stokes theorem the conserved charge can now be expressed as

$$Q(\xi) = \frac{1}{8\pi G_3} \int \mathcal{F}^{\mu\nu} dS_{\mu\nu}, \tag{4.8}$$

where S is the boundary of Σ .

We now examine a specific example, namely the BTZ black hole that is asymptotic to the supersymmetric AdS₃ vacuum. The solution is given by [26]

$$ds^2 = -U dt^2 + \frac{dr^2}{U} + r^2(d\phi - \frac{4J}{r^2} dt)^2, \quad U = r^2 - 8M + \frac{16J^2}{r^2}. \tag{4.9}$$

From now on we set G_3 to be 1. One may define the mass E and angular momentum L as the conserved charges associated with the Killing vectors $K_t = \partial/\partial t$ and $K_\phi = \partial/\partial \phi$ respectively. It is straightforward to work out these quantities, given by

$$E = \left(\sigma + \frac{1}{2\nu^2}\right)M - \frac{J}{\mu}, \quad L = \left(\sigma + \frac{1}{2\nu^2}\right)J - \frac{M}{\mu}. \tag{4.10}$$

Note that the second and third brackets in (4.7) converges too fast to give any contributions.

To validate the above mass and angular momentum, we examine the first law of thermodynamics. The temperature and the angular velocity can be obtained directly from the metric, given by

$$T = \frac{r_+^4 - 16J^2}{2\pi r_+^3}, \quad \Omega = \frac{4J}{r_+^2}, \tag{4.11}$$

where

$$r_\pm = \sqrt{2(M+J)} \pm \sqrt{2(M-J)}. \tag{4.12}$$

The entropy can be obtained using the Cardy formula *via* the AdS/CFT correspondence, given by [5]

$$S = \frac{1}{3}\pi^2 C_L T_L + \frac{1}{3}\pi^2 C_R T_R, \tag{4.13}$$

where the central charges $C_{L,R}$ are given by (3.34) and $T_{L,R}$ are given by

$$T_L = \frac{r_+ - r_-}{2\pi}, \quad T_R = \frac{r_+ + r_-}{2\pi}. \quad (4.14)$$

It is now straightforward to verify the first law of thermodynamics, namely

$$dE = TdS + \Omega dL. \quad (4.15)$$

The result should not be surprising since the effect of ν in thermodynamical quantities is to shift the parameter σ uniformly.

For the BTZ black hole, we have $M \geq |J|$; nevertheless, the mass can be negative for generic parameters (σ, ν^2, μ) . However, recall that the parameter conditions for the critical points (3.35), we have

$$E_{\text{crit}} = \frac{M}{|\mu|} - \frac{J}{\mu}, \quad L_{\text{crit}} = \frac{J}{|\mu|} - \frac{M}{\mu}. \quad (4.16)$$

Thus we see that at the critical points, the mass is non-negative. Furthermore the quantity $(E_{\text{crit}} - L_{\text{crit}})$ is either positive for $\mu > 0$ or 0 for $\mu < 0$, but never negative.

It is worth pointing out that since the BTZ black hole does not involve the scalar multiplet, its mass calculation is the same for that in pure massive supergravity. Our results thus demonstrate the negativeness of the mass of the BTZ black for pure massive supergravity, which was not studied previously. The effect of adding scalar multiplet provides an additional constraint to the parameters. This can be seen more clearly by considering the case where the Lorentz-Chern-Simons term is turned off, by setting $\mu \rightarrow \infty$. In this case at the critical point, there is no massive graviton and the BTZ black hole has zero energy and angular momentum. Thus for pure gravity, the sign choice of σ can be either positive or negative. The inclusion of the scalar multiplet will force that σ to be positive.

5 BPS string solution

5.1 The solution

In this section, we construct BPS string solutions with $R^{1,1}$ isometry. The ansatz is given by

$$ds^2 = dz^2 + e^{2A(z)}(-dt^2 + dx^2), \quad \phi = \phi(z), \quad S = S(z). \quad (5.1)$$

(Such a metric ansatz is also called domain wall.) It is worth pointing out immediately that for this metric ansatz, the \mathcal{L}_{LCS} term gives no contribution to the equations of motion. A natural choice for the vielbein is given by

$$e^0 = e^A dt, \quad e^1 = e^A dx, \quad e^2 = dz. \quad (5.2)$$

The non-vanishing components of the spin connection and curvature are

$$\begin{aligned}\omega_{02} &= -A' e^0, & \omega_{12} &= A' e^1, \\ R_{0101} &= A'^2, & R_{0202} &= A'' + A'^2, & R_{1212} &= -R_{0202},\end{aligned}\quad (5.3)$$

Here a prime denotes a derivative with respect to z . The three-dimensional gamma matrices can be chosen to be

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.4)$$

It follows from (2.3) that a Killing spinor ϵ satisfies

$$D_\mu \epsilon + \frac{1}{2} \gamma_\mu S \epsilon = 0, \quad (5.5)$$

$$(\gamma^\mu \partial_\mu \phi + S + m) \epsilon = 0. \quad (5.6)$$

From (5.6), we deduce that

$$(\partial \phi)^2 = (S + m)^2, \quad (5.7)$$

Thus for our string ansatz, we have, without loss of generality,

$$\phi' = -(S + m), \quad (5.8)$$

This corresponds to the following projection

$$(\gamma_2 - 1) \epsilon = 0 \quad \Rightarrow \quad \epsilon = \begin{pmatrix} \chi \\ 0 \end{pmatrix}. \quad (5.9)$$

The integrability condition for (5.5) is

$$\frac{1}{4} R_{\alpha\beta\nu\mu} \gamma^{\alpha\beta} \epsilon + \frac{1}{2} (\gamma_\mu \partial_\nu S - \gamma_\nu \partial_\mu S) \epsilon + \frac{1}{2} \gamma_{\nu\mu} S^2 \epsilon = 0. \quad (5.10)$$

Substituting the Riemann curvature (5.3) and the killing spinor (5.9), we find that the only solution is

$$S = -A'. \quad (5.11)$$

The Killing spinor can then be solved explicitly from (5.5), given by

$$\epsilon = \begin{pmatrix} e^{\frac{1}{2}A} \\ 0 \end{pmatrix} \quad (5.12)$$

Under the string ansatz, the ϕ equation (2.4) becomes

$$(8m + 2S + 6A')(A' + S) + 4(S + A')' = 0. \quad (5.13)$$

It is automatically satisfied by (5.11). The S equation (2.5) can be simplified and becomes

$$(\alpha + 8b)S'' + 2(4b + 3c)SS' - 2aS' + cS^3 + \sigma e^{-2\phi}S + \tilde{m} = 0. \quad (5.14)$$

The Einstein equations of motion (2.6) are then all automatically satisfied.

It is now straightforward to obtain the supersymmetric AdS vacuum solution, given by

$$\sigma = \frac{\tilde{m} - cm^3}{m}, \quad S = -m, \quad \phi = 0, \quad R_{ij} = -2m^2 g_{ij}. \quad (5.15)$$

Here we let σ be continuous so that ϕ is zero instead of a non-vanishing constant. The general solution for (5.14) is not expected to be solved explicitly. For certain choices of the parameters, explicit solutions can be obtained. Let us first consider the case with

$$(\alpha, a, b, c) = 0. \quad (5.16)$$

The solution is given by

$$\begin{aligned} ds^2 &= \frac{e^{2mz}}{1 + qe^{2mz}}(-dt^2 + dx^2) + dz^2, \\ e^{-2\phi} &= \frac{\tilde{m}}{m}(1 + qe^{2mz}), \quad S = -\frac{m}{1 + qe^{2mr}}. \end{aligned} \quad (5.17)$$

The coordinate z runs from $-\infty$ to $+\infty$, and the metric interpolates between the AdS_3 horizon and asymptotic flat Minkowski space-time. The string coupling constant $g = e^\phi$ runs from the constant $\sqrt{m/\tilde{m}}$ to 0 at the asymptotic flat region. The solution can be lifted to $D = 6$ and it becomes a dyonic string with the “1” in the harmonic function associated with the magnetic component dropped. To be specific, we have

$$\begin{aligned} ds_6^2 &= H_e^{-1}(-dt^2 + dx^2) + H_m(dr^2 + r^2 d\Omega_3^2), \\ e^{-2\phi} &= \frac{\tilde{m}}{m} \frac{H_m}{H_e}, \quad H_{(3)} = dx \wedge dt \wedge dH_e^{-1} + m^2 \Omega_{(3)}, \\ H_e &= 1 + \frac{q}{r^2}, \quad H_m = \frac{m^2}{r^2}. \end{aligned} \quad (5.18)$$

The second case we would like to consider is the following

$$a = 0, \quad b = -\frac{3}{4}c, \quad \alpha = 6c, \quad \tilde{m} = 0. \quad (5.19)$$

We find the solution is given by

$$\begin{aligned} ds^2 &= (e^{mz} - q)^2(-dt^2 + dx^2) + dz^2, \\ S &= -\frac{m}{1 - qe^{-mz}}, \quad e^{-2\phi} = -\frac{cm}{(1 - qe^{-mz})^2}. \end{aligned} \quad (5.20)$$

In this case the metric approaches AdS_3 at the asymptotic $z \rightarrow \infty$ and has a naked singularity in the middle.

The above two solutions can be grouped together with the parameter choice (3.1). In this case, it is advantageous to make a coordinate transformation and treat S as the radial coordinate. The solution is then given by

$$\begin{aligned} ds^2 &= e^{2A}(-dt^2 + dx^2) + \frac{dS^2}{S^2 f^2}, \\ e^{2\phi} &= \frac{S}{S^3 - S_2^3}, \quad A = \int \frac{1}{f} dS \quad f = -\frac{2(S - S_1)(S^3 - S_2^3)}{S_2^3 + 2S^3}, \end{aligned} \quad (5.21)$$

where

$$S_1 = -m < 0, \quad S_2 = -(6\nu^2 \tilde{m})^{1/3} < 0. \quad (5.22)$$

The explicit expression for A is somewhat complicated, given by

$$\begin{aligned} A &= A_0 + \frac{1}{4(S_1^3 - S_2^3)} \left(-2\sqrt{3}S_1 S_2 (S_1 - S_2) \arctan\left(\frac{2S + S_2}{\sqrt{3}S_2}\right) \right. \\ &\quad - 2(2S_1^3 + S_2^3) \log(S - S_1) + 2S_1 S_2 (S_1 + S_2) \log(S - S_2) \\ &\quad \left. - S_1 S_2 (S_1 + S_2) \log(S^2 + S S_2 + S_2^2) + 2S_2^3 \log(S^3 - S_2^3) \right), \end{aligned} \quad (5.23)$$

where A_0 is an integration constant and it should be chosen appropriately such that the expression for A is real.

In the vicinity of $S = 0$, the metric describes Minkowski space-time. In the vicinity of $S = S_1 \equiv -m < 0$, the solution approaches the vacuum AdS_3 . In the vicinity of $S = S_2 \equiv -(\tilde{m}/c)^{1/3} < 0$, the solution becomes

$$ds^2 \sim e^{-2S_2 z} (-dt^2 + dx^2) + dz^2, \quad \phi \sim (S_1 - S_2)z. \quad (5.24)$$

This linear-dilaton solution is approximate, valid for $z \rightarrow \infty$ when $S_2 < S_1$, and for $z \rightarrow -\infty$ when $S_2 > S_1$. For both cases, the “string” coupling $g = e^\phi$ goes to infinity. It is clear that in the region $S \in (\max(S_1, S_2), 0)$, the metric interpolates between the flat Minkowski space-time and the horizon of an AdS_3 . For $S_1 < S_2 < 0$, the region of $S \in (S_1, S_2)$ describes an interpolation between the boundary of the vacuum AdS_3 and the AdS_3 with the linear dilaton. To see this, we note that the function f is negative in this region and that

$$A = \int_{S_0}^S \frac{1}{f} dS, \quad (5.25)$$

where $S_1 < S_0 < S_2$. Thus $A(S_1) \rightarrow +\infty$ and $A(S_2) \rightarrow -\infty$, indicating a boundary and a horizon structure respectively. For $S_2 < S_1 < 0$, the role the two AdS_3 reverses, and the horizon lies in the vacuum AdS_3 whilst the boundary lies in the AdS_3 with the linear dilaton.

5.2 Spectrum analysis

In the previous subsection, we discuss the general BPS string solutions. For certain parameter choice, we obtain explicit solutions. Some of these metrics are asymptotic to AdS_3 and hence are dual to certain two-dimensional field theory that has an ultra-violet conformal fix points. In the general discussion of the AdS/CFT correspondence, the correlation functions and the spectrum of the corresponding strongly coupled two-dimensional theory can be analyzed by studying the wave equations in these gravitational backgrounds in the Einstein frame. Making appropriate coordinate transformations, the metric can be cast into the following form.

$$ds^2 = e^{2\tilde{A}}(-dt^2 + dx^2 + dr^2), \quad (5.26)$$

The simplest two-point function is that of the operator $\mathcal{O} \sim \text{tr} F^2$, which is expected to be coupled to a massless s -wave free scalar ϕ , satisfying

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0. \quad (5.27)$$

For the s -wave, we have $\phi = e^{i\omega t}e^{-\frac{1}{2}\tilde{A}}\chi(r)$, where ω measures the energy level of the solution. It is easy to show that χ satisfies Schrödinger equation

$$\left[-\partial_r^2 + V(r)\right]\chi = \omega^2\chi, \quad (5.28)$$

where

$$V = \frac{1}{4}(2\tilde{A}'' + \tilde{A}'^2). \quad (5.29)$$

Note that the potential can be written as $V = U^2 + U'$ where the superpotential is $U = \frac{1}{2}\tilde{A}'$, thus this is a supersymmetric quantum mechanics system [27]. We now consider the solution (5.20). First we convert the metric to the Einstein frame by a conformal rescaling of the metric $ds^2 \rightarrow e^{4\phi}ds^2$. We can then cast the resulting metric into the conformal form (5.26), with

$$e^{2\tilde{A}} = \frac{e^{6mqr}}{(1 - e^{mqr})^2}. \quad (5.30)$$

Thus the potential V is

$$V = \frac{m^2q^2(4e^{2mqr} - 10e^{mqr} + 9)}{4(1 - e^{mqr})^2}. \quad (5.31)$$

The variable r runs from $-\infty$ to 0. In this region, we have $\frac{4}{9}m^2q^2 \leq V < \infty$. Thus, it is clear that the spectrum is continuous with a mass gap $V_{\min} = \frac{9}{4}m^2q^2$.

The structure for the solution (5.21) is more complicated, and the explicit form of the potential V cannot be obtained. The characteristics of the potential can nevertheless be

analyzed. Although in the string frame, the metric runs from an AdS_3 horizon to an AdS_3 boundary, in the Einstein frame, one of the AdS_3 turns into a singular metric owing to the linear dilaton. For the theory to dual to a two-dimensional quantum field theory, it is easier to consider the case with $S_1 < S_2$ so that its asymptotic behavior is AdS_3 with a naked singularity in the bulk. Following the same procedure outlined for the previous simpler example, we obtain that now r lies in the region (r_1, r_2) with the potential V behaves as follows

$$\begin{aligned} r \rightarrow r_2 \quad V &\sim \frac{3}{4(r-r_2)^2}, \\ r \rightarrow r_1 \quad V &\sim \frac{(3S_2 - 2S_1)(5S_2 - 2S_1)}{4S_2^2(r-r_1)^2}. \end{aligned} \quad (5.32)$$

Thus in general the system has a discrete spectrum.

6 General supersymmetric solutions

By definition, in any supersymmetric background, there exists a solution of the Killing spinor equations (5.5, 5.6). From the Killing spinor ϵ , we can construct a Killing vector

$$K^\mu = \bar{\epsilon} \gamma^\mu \epsilon, \quad (6.1)$$

which is null. It follows from (5.5) that we have

$$S K^\lambda = \frac{\epsilon^{\lambda\alpha\beta}}{2\sqrt{-g}} \partial_\alpha K_\beta, \quad \epsilon^{012} = 1. \quad (6.2)$$

As was demonstrated in [14], the general metric ansatz with a null Killing vector $K = \partial/\partial_v$ can be casted into the following form

$$ds^2 = e^{2A} dz^2 + e^{2B} du^2 + 2e^{2mz} dudv, \quad (6.3)$$

where the functions A and B depend on the coordinates (u, z) . The non-vanishing components of the Ricci tensor are given by

$$\begin{aligned} R_{uu} &= -\frac{1}{2}e^{-2A}[h'' - (2m + A')h' + 4m^2h] - (\ddot{A} + \dot{A}^2), \\ R_{uv} &= m(A' - 2m)e^{2(mz-A)}, \quad R_{uz} = m\dot{A}, \quad R_{zz} = 2m(A' - m), \end{aligned} \quad (6.4)$$

where a prime and a dot denote a derivative with respect to z and u respectively and $h = e^{2B}$. The Ricci scalar is given by

$$R = e^{-2A}(4mA' - 6m^2). \quad (6.5)$$

It follows from (6.2) that we have the following algebraic constraint

$$S = -me^{-A}. \quad (6.6)$$

Substituting this relation into (5.5), we can find that the killing spinor is given by

$$\epsilon = \begin{pmatrix} e^{-\frac{1}{2}B+ mz} \\ 0 \end{pmatrix} \quad (6.7)$$

Comparing this to the Killing spinor (5.12) in the earlier section, one can deduce that when $B = mz$, the general solutions reduce to the previous simpler string solution.

Since $g^{uu} = 0$, it follows from (5.6) that we have $e^{-A}\phi' = \pm(S + m)$. The \pm sign choices are inequivalent. The plus-sign choice leads only to $S = -m$ and constant ϕ , which we shall discuss later. For the minus-sign, *i.e.*

$$e^{-A}\phi' = -(S + m), \quad (6.8)$$

it is easy to verify that the dilaton equation (2.4) is satisfied by (6.8), together with (6.6). The S equation of motion (2.5) now becomes

$$(\alpha + 8b)S(S^2)'' + 4amSS' - 4(4b + 3c)mS^2S' + 2m^2(cS^3 + e^{-2\phi}S + \tilde{m}) = 0. \quad (6.9)$$

The Einstein equations of motion are generally satisfied by (6.6, 6.8, 6.9) except for one in the (u, u) direction. This equation is effectively a 4'th-order linear differential equation, but it is rather complicated and we shall not present it here.

It is worth pointing out that the three equations (6.6, 6.8, 6.9) are self-contained involving three functions (A, ϕ, S) and derivatives on coordinate z . Thus these can be viewed as ordinary differential equations with the integration constant depending on the coordinate u . The Einstein equation then determines the metric function $B(u, z)$.

Let us consider a simple case with $(a, b, c, \alpha) = 0$. The functions (A, ϕ, S) can be solved explicitly, given by

$$e^{-2\phi} = -\frac{\tilde{m}}{S}, \quad e^A = -\frac{m}{S}, \quad S = -(m + q(u)e^{2mz}). \quad (6.10)$$

The Einstein equation implies that

$$\begin{aligned} & \beta h''' + \frac{1}{2S^2} \left(6\beta mS(2m + S) - \tilde{m} \right) h'' - \frac{m}{S^2} \left(2\beta m(3S^2 + 2mS - 2m^2) - \tilde{m} \right) h' \\ & - \frac{8\beta m^3(m + S)^2}{S^2} h + \frac{m^2}{S^6} \left(4\beta m^2 S(S\ddot{S} - \dot{S}^2) + \tilde{m}(2S\ddot{S} - 3\dot{S}^2) \right) = 0, \end{aligned} \quad (6.11)$$

where $h(u, z) = e^{2B}$. This can be viewed as an ordinary linear differential equation for h with a source, but with integration constants now being arbitrary functions of u .

The general explicit solutions for the special case with $q(u) = 0$ was obtained in [14]. Alternatively if we set $\beta = 0$, corresponding to turning off the Lorentz Chern-Simons term, the general explicit solution also exists, given by

$$e^{2B} = h = f_1(u) + e^{2mz} f_2(u) - \frac{\dot{S}^2}{4(m+S)^2 S^2} + \left(\frac{m - 4mzS - 4zS^2}{2m^2(m+S)S} + \frac{\log(-S)}{m^3} \right) \ddot{S}. \quad (6.12)$$

Here f_1 and f_2 are two arbitrary functions of u . It is clear that f_2 can be absorbed by a gauge transformation $v \rightarrow v - \frac{1}{2} \int f_2 du$.

We now consider the parameter choice (3.1). In this case, we have

$$e^{-2\phi} = -\frac{cS^3 + \tilde{m}}{S}, \quad e^A = -\frac{m}{S}, \quad (6.13)$$

where S satisfies

$$S' = f \equiv \frac{2m(S+m)(cS^3 + \tilde{m})}{\tilde{m} - 2cS^3}. \quad (6.14)$$

This can be solved as follows

$$z + y(u) = \int \frac{dS}{f}, \quad (6.15)$$

where y is an arbitrary function of u . This implies that

$$dz + dy = \frac{dS}{f}, \quad \dot{S} = f\dot{y}. \quad (6.16)$$

Thus we may chose (S, y) as coordinates to replace the original (z, u) . The Einstein equation gives rise to a linear differential equation with only z derivative on $h = e^{2B}$ up to the 4'th order. The detail expression is rather complicated and we shall not present here. The result is simplified significantly if we further set $\tilde{m} = 0$. In this case, S can be solved explicitly, given by

$$S = -(m + q(u)e^{-mz}), \quad (6.17)$$

where q is an arbitrary function of u . The function $h = e^{2B}$ satisfies the following equation

$$\begin{aligned} & ch'''' + \frac{m}{3S} (\beta m - 6c(3m + 5S)) h''' \\ & + \frac{m^2}{3S^2} (-3\beta m(m + 2S) + c(21m^2 + 117mS + 109S^2)) h'' \\ & + \frac{m^3}{3S^3} (\beta m(m^2 + 11mS + 12S^2) - c(3m^3 + 81m^2S + 246mS^2 + 170S^3)) h' \\ & + \frac{2m^4(m + S)}{3S^3} (-\beta m(m + 4S) + 3c(m^2 + 12mS + 16S^2)) h \\ & + \frac{2m^4}{3S^6} (\beta m^2(4\dot{S}^2 - S\ddot{S}) + c(24m^2\dot{S}^2 - 6mS\dot{S}^2 - 3m^2S\ddot{S} + 6S^2\dot{S}^2 - 2S^3\ddot{S})) = 0. \end{aligned} \quad (6.18)$$

Since there is only z derivatives on h , this is effectively an fourth-order ordinary linear differential equation with a source. The integration constants should be considered as arbitrary functions of u .

A special class of pp-wave solution corresponds to setting

$$S = -M, \quad \phi = 0, \quad A = 0. \quad (6.19)$$

The S equation (2.5) then requires that $\tilde{m} = m + cm^3$. The Einstein equations reduce to the following differential equation

$$\begin{aligned} \alpha h'''' - 2(2\alpha + \beta)m h''' + (1 + 2am + (4\alpha + 6\beta + 3c)m^2)h'' \\ - 2m(1 + 2am + (2\beta + 3c)m^2)h' = 0. \end{aligned} \quad (6.20)$$

The solution is given by

$$h = f_1(u) + f_2(u)e^{2mz} + f_3(u)e^{c+z} + f_4(u)e^{c-z}, \quad (6.21)$$

where f_i 's are four arbitrary functions of u and constants are given by

$$c_{\pm} = \alpha^{-1} \left[(\alpha + \beta)m \pm \sqrt{(\alpha^2 + \beta^2 - 3\alpha c)m^2 - \alpha(1 + 2am)} \right]. \quad (6.22)$$

It is clear that when $\alpha = \beta = c = a = 0$, the solution reduces to the one given in [14] for the original topologically massive supergravity. There are two additional classes of special solutions. The first corresponds to having $c = -(1 + 2am + 2\beta m^2)/(3m^2)$ such that $c_+ = 0$ or $c_- = 0$. The function h is given by

$$h = f_1(u) + f_2(u)e^{2mz} + f_3(u)z + f_4(u)e^{2mz(1+\beta/\alpha)}. \quad (6.23)$$

The second class corresponds to have $c = -(1 + 2am - 2\alpha m^2)/(3m^2)$ and $\beta = -\alpha$, for which $c_+ = 0 = c_-$. The function h is given by

$$h = f_1(u) + f_2(u)e^{2mz} + f_3(u)z + f_4(u)z^2. \quad (6.24)$$

In all of the above solutions, the terms of f_1 and f_2 are pure gauge [14]. Note that the coordinate z is a logarithmic function of the global radial coordinate. Specializing the parameters to the theory discussed in section 2, we find that the first case in the above corresponds to the critical conditions. Thus logarithmic modes can emerge at the critical points and to understand them *via* a logarithmic CFT [28], one should choose a boundary condition [29] that is less restrictive than that advocated in [30].

7 Conclusions

In this paper we study generalized topologically massive supergravity that was recently constructed in [11]. The theory is hybrid in a sense that it consists one off-shell supergravity multiplet and one on-shell matter scalar multiplet. An important feature of three-dimensional massive supergravities is that the supersymmetry can be realised off shell. As

a consequence, such a theory can be complete in terms of supersymmetry by augmenting with only a finite number of higher-derivative terms. The hybrid theory studied in this paper implies that the matter sector can still be minimally dynamical with at most two derivatives even though the higher derivative terms in gravity sector is inevitable in a quantum theory. For generic parameters, the auxiliary scalar field in the off-shell supergravity multiplet is dynamical. We focus on a special class of parameter choices for which the auxiliary field is non-dynamical. The resulting theory has five parameters and it can be viewed as generalized super NMG theory with a matter scalar multiplet.

Since the super NMG theory with negative Einstein-Hilbert action can be ghost free, we analyze the linear perturbation of our generalized super NMG theory in Minkowski space-time. We find that the spectrum contains one massless scalar mode and two massive graviton modes. For general parameters, when the Einstein-Hilbert action is negative, the two graviton modes are ghost free, but the scalar mode is ghost like. When the action is positive, the scalar mode is non-ghost, but one of the two massive graviton is ghost like. However, we find a special choice of parameters such that the on-shell Hamiltonian for both massive graviton vanish. This signals that the linearized analysis breaks down and it suggests a possibility that the theory may become ghost free. A proper higher-order analysis is relegated for future research.

Next, we perform linearized analysis around the supersymmetric AdS_3 vacuum. For generic parameters, the theory contains one scalar mode and two non-trivial massive gravitons. For the scalar mode to be ghost free, it is necessary that the coefficient σ of the super invariant involving the Einstein-Hilbert term is positive. Then one of the two massive gravitons becomes inevitably ghost like. We obtain the critical conditions for which the ghost graviton becomes pure gauge and decouples from the bulk physics. We then show that the remaining massive graviton can indeed have positive energy. (This conclusion also applies to the corresponding pure massive supergravity (3.5).) Furthermore, we also demonstrate explicitly that the scalar mode is stable satisfying the Breitlohner-Freedman bound. These properties suggest that the theory may be well-defined at these critical points. In order to establish this point further, we obtain the mass and angular momentum for the BTZ black hole that is asymptotic to the AdS_3 . We find that indeed at the critical points, the mass is non-negative and furthermore it is always greater or equal to the angular momentum. We also verify explicitly that the first law of thermodynamics holds.

It should be emphasized that although our analysis focused on the hybrid theory, the linearized analysis of the traceless graviton mode and the BTZ energy calculation apply

equally well to the four-parameter pure massive gravity. The critical points for the general massive pure supergravity were obtained in [8]. Our further examination of the energy of the remaining non-trivial massive graviton and the mass of the BTZ black hole reveals additional properties of the theory at these critical points. The inclusion of the on-shell matter multiplet does not alter these results, but instead provides further constraints on the parameter space.

We also construct BPS solutions of the general theory. We find that the equations are reduced to effectively two linear differential equations of two functions. For some specific parameter choices, the solutions can be solved explicitly. In particular, we obtain two types of exact solutions. One is the pp-wave propagating in the AdS_3 background, including the one arising at the critical points. The other is the BPS string (domain wall) solution, which is dual to some two-dimensional boundary theory with an ultra-violet conformal fixed point. We obtain the characteristics of the spectrum using the standard free-scalar analysis.

Our results are the first tentative approach to understand a possible quantum supergravity in three dimensions with inevitable higher derivatives terms in the supergravity sector, but with the standard dynamics in the matter sector. The existence of a remaining well-defined massive graviton at the critical points makes bulk gravity non-trivial, giving rise to a more interesting model to study quantum gravity. Although the scalar matter multiplet we considered is a rather simple example of a broad class of matter couplings without higher derivatives (in contrast with string theory), our results nevertheless reveals some general features of these theories. It might be possible to construct a quantum supergravity with inevitable but finite number of higher-derivative terms in the gravity sector coupled with a matter sector of the standard dynamics.

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